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Prediction Operators in Banach Ideal Spaces[1]

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A necessary and sufficient condition for an operator in a Banach ideal space to be a prediction operator is given. © 2001 Elsevier Science (USA)

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Let X be a Banach space, C a set in X, and $x \in X$. An element $y \in C$ is called an element of best approximation of x in C if we have

$$\|x-y\| = \inf\{\|x-z\| : z \in C\}.$$

If the element of best approximation of x in C is unique, we denote it by $\pi(x \mid C)$. The operator $\pi(\cdot \mid C)$ is called the best approximation operator of X onto C.

A space X is said to be strictly convex if for each pair $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$ we have $\|x+y\| < 1$. The space X is said to be smooth if at each point $0 \neq x \in X$, there is only one support functional f_x, i.e., $f_x \in X^*$, $\|f_x\|_{X^*} = 1$, and $f_x(x) = \|x\|$.

Let (G, Σ, μ) be a σ-finite measure space and denote by $S = S(G, \Sigma, \mu)$ the set of all equivalence classes of Σ-measurable real-valued functions on G with algebraic operations and order defined in a natural way. A linear subset X of S endowed with some norm $\|\cdot\|$ is called a Banach ideal space if $(X, \|\cdot\|)$ is complete and it satisfies the condition that $x \in X$, $y \in S$, and

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$|y| \leqslant |x|$ imply $y \in X$ and $\|y\| \leqslant \|x\|$. The norm on a Banach ideal space is called order-continuous if in addition it satisfies the condition that $x_n \downarrow 0$ implies $\|x_n\| \to 0$. It is well known that $L^p(1 \leqslant p \leqslant \infty)$, Orlicz spaces, Lorentz spaces, Orlicz–Lorentz spaces, and Musielak–Orlicz spaces are all Banach ideal spaces. Criterions of order-continuity, strict convexity, smoothness, and reflexivity of these spaces can be found in the literature.

For a Banach ideal space $X(G, \Sigma, \mu)$ (which is simply denoted by $X(\Sigma)$), let Σ' be a σ-sublattice of the σ algebra Σ and $X(\Sigma') = \{x \in X(\Sigma) : x \text{ is } \Sigma' \text{ measurable}\}$. If $\pi(\cdot \mid X(\Sigma'))$ exists, it is called a prediction operator. Prediction operators have wide applications in probability, Bayes estimation theory, prediction theory, and many other fields. Various authors have studied these in the past 30 years [2–12]. For example, Dykstra has studied the case L^2 in [4]. Landers and Rogge have studied the case L^p in [6] and then obtained a result of Orlicz space L_M for the modular in [5]. Duan and Chen gave a necessary condition (which is not sufficient) and a sufficient condition (which is not necessary) for prediction operators in L_M in [11]. In 1995, Wang *et al.* [12] obtained a necessary and sufficient condition for an operator in L_M to be a prediction operator. The purpose of this paper is to generalize the main result in [12] to Banach ideal spaces.

THEOREM. *Let $X(\Sigma)$ be a reflexive, strictly convex, and smooth Banach ideal space. Then the operator $T: X(\Sigma) \to X(\Sigma)$ is a prediction operator (i.e., there exists σ-sublattice $\Sigma' \subset \Sigma$ such that $T(\cdot) = \pi(\cdot \mid X(\Sigma'))$ if and only if it satisfies the following conditions:*

(i) $T(Tx) = Tx (x \in X(\Sigma))$;

(ii) $r \in TX$ *if* $r \in R$;

(iii) $x, y \in TX$ *and* $\alpha, \beta \in R^+$ *imply* $\alpha x + \beta y \in TX$;

(iv) $\{x_n\} \subset TX$ *and* $\|x_n - x\| \to 0 (n \to \infty)$ *imply* $x \in TX$;

(v) $x \in TX$ *and* $r \in R$ *imply* $x \vee r, x \wedge r \in TX$;

(vi) $\chi_A \in TX$ *and* $\alpha, \beta \in R^+$ *and* $r \in R$ *imply* $\|x - Tx\| \leqslant \|x - \alpha Tx - \beta \chi_A + r\|$,

where R is the set of all real numbers, R^+ is the set of non-negative numbers, and χ_A is the characteristic function of the set A.

Proof. Necessity. Suppose Σ' is a σ-sublattice of Σ and $T(\cdot) = \pi(\cdot \mid X(\Sigma'))$. Obviously, $TX = X(\Sigma')$, so (i) is clear.

Note that $r \in R$ implies $r \in X(\Sigma')$ since $\phi, G \in \Sigma'$, so (ii) is true.

It is easy to find that $X(\Sigma')$ is a closed convex cone in $X(\Sigma)$ since Σ' is a σ-sublattice. So (iii) and (iv) are true.

For brevity, we use the notation $\{x > a\}$ for the inverse image of the set (a, ∞) under the mapping $x: G \to R$. Since

$$\{x \vee r > a\} = \begin{cases} G & (a < r) \\ \{x > a\} & (a \geqslant r) \end{cases}$$

and

$$\{x \wedge r > a\} = \begin{cases} \{x > a\} & (a < r) \\ \phi & (a \geqslant r), \end{cases}$$

we see that $x \vee r$ and $x \wedge r$ are Σ'-measurable, and hence $x \vee r$, $x \wedge r \in X(\Sigma') = TX$. This establishes (v).

Since $Tx \in TX$, $\chi_A \in TX$, $r \in TX$, and TX is a convex cone, we have $\alpha Tx + \beta \chi_A - r \in TX$. Therefore $\|x - Tx\| \leqslant \|x - (\alpha Tx + \beta \chi_A - r)\|$, and so (vi) holds.

Sufficiency. Let $\Sigma' = \{A \in \Sigma : \chi_A \in TX\}$. Divide the proof into three steps as follows:

1. We prove that Σ' is a σ-sublattice of Σ.

By (ii) and (i), $\chi_\Phi = 0 = T0 = T\chi_\Phi$, $\chi_G = 1 = T1 = T\chi_G$. So $\chi_\Phi, \chi_G \in TX$, that is, $\Phi, G \in \Sigma'$. If $A, B \in \Sigma'$, then $\chi_A, \chi_B \in TX$. Observe that $\chi_{A \cup B}(t) = (\chi_A(t) + \chi_B(t)) \wedge 1$ and $\chi_{A \cap B}(t) = (\chi_A(t) + \chi_B(t) - 1) \vee 0$. Thus, by (iii) and (v), $\chi_{A \cup B}, \chi_{A \cap B} \in TX$. So $A \cup B$, $A \cap B \in \Sigma'$.

Let $A = \bigcup_{n=1}^{\infty} A_n$, where $A_n \in \Sigma' (n = 1, 2, \ldots)$ and $A_1 \subset A_2 \subset A_3 \subset \cdots$. By Theorem 10 (Ogasawara) in [1, Chap. 10, Sect. 4] we find that $X(\Sigma)$ is order-continuous since $X(\Sigma)$ is reflexive. Therefore $\|\chi_A - \chi_{A_n}\| \to 0$ as $n \to \infty$ because $\mu(A/A_n) \to 0$ as $n \to \infty$. Using (iv), $\chi_A \in TX$ and it follows $A \in \Sigma'$. If $A = \bigcap_{n=1}^{\infty} A_n$ where $A_n \in \Sigma' (n = 1, 2, \ldots)$ and $A_1 \supset A_2 \supset A_3 \supset \cdots$. By the above same arguments we can deduce that $A \in \Sigma'$. Thus we have proved that Σ' is a σ-sublattice of Σ.

2. We prove that $X(\Sigma') = TX$.

Suppose $x \in TX$. For any $a \in R$ we write $y_n(t) = ((3/2)^n (x(t) - a) \vee 0) \wedge 1$. By (iii) and (v) we see that $y_n \in TX (n = 1, 2, \ldots)$. Observe that $x(t) \leqslant a$ implies $y_n(t) = 0 = \chi_{\{x-a\}}(t)$ and $x(t) > a$ implies $y_n(t) \to 1 = \chi_{\{x > a\}}(t)$ $(n \to \infty)$. We can deduce $\chi_{\{x > a\}} - y_n \downarrow 0$ $(n \to \infty)$ and so $\|\chi_{\{x > a\}} - y_n\| \to 0$ $(n \to \infty)$ by the order-continuity of $X(\Sigma)$. It follows that $\chi_{\{x > a\}} \in TX$ from (iv). That is, $\{x > a\} \in \Sigma'$ and then $x \in X(\Sigma')$. Hence $TX \subset X(\Sigma')$.

Conversely, suppose $x \in X(\Sigma')$. Then $\{x > a\} \in \Sigma'$ and so $\chi_{\{x > a\}} \in TX$ for any $a \in R$.

Notice that $x \vee (-m) \downarrow x$ as $m \to \infty$ and thus $\|x \vee (-m) - x\| \to 0$ as $m \to \infty$. We need only prove $x \vee (-m) \in TX (m = 1, 2, \ldots)$ to show that $x \in TX$ by (iv). However, $x \vee (-m) = (x \vee (-m) + m) - m$ and $-m \in TX$, so from (iii) we see that we only need to prove $x \vee (-m) + m \in TX$. Moreover, $x \vee (-m) + m \geqslant 0$ and $x \vee (-m) + m \in X(\Sigma')$, since $X(\Sigma')$ is a convex cone. In what follows, we need only prove that $0 \leqslant x \in X(\Sigma')$ implies $x \in TX$.

By Lemma 3 in [1, Chap. 4, Sect. 3],

$$\left\| x - \sum_{k=0}^{n2^n} (k/2^n)\, \chi_{\{(k+1)/2^n \geqslant x \geqslant k/2^n\}} \right\| \to 0 \qquad \text{as} \quad n \to \infty.$$

However,

$$\sum_{k=0}^{n2^n} (k/2^n)\, \chi_{\{(k+1)/2^n \geqslant x \geqslant k/2^n\}} = (1/2^n) \sum_{k=0}^{n2^n} \chi_{\{x > k/2^n\}} \in TX$$

since $\chi_{\{x > k/2^n\}} \in TX (n = 1, 2, ...; k = 1, 2, ..., n2^n)$ and by using (iii). Consequently $x \in TX$ by (iv). Hence $X(\Sigma') \subset TX$.

3. We prove that $T(\cdot) = \pi(\cdot \mid X(\Sigma'))$.

It is well known that $\pi(\cdot \mid X(\Sigma'))$ has meaning since $X(\Sigma)$ is reflexive and strictly convex.

If $x \in TX$, then $Tx = x = \pi(x \mid X(\Sigma'))$.

Now suppose $x \notin TX$. Since $X(\Sigma)$ is smooth, $x - Tx$ has a unique support functional $f_{x-Tx} \in X(\Sigma)^*$ such that $\|f_{x-Tx}\|_* = 1$ and $f_{x-Tx}(x - Tx) = \|x - Tx\|$. Also $f_{x-Tx}(Tx) = \lim_{\lambda \to 0} (\|x - Tx + \lambda Tx\| - \|x - Tx\|)/\lambda$ exists in the Gateaux sense. Nevertheless, $\|x - Tx + \lambda Tx\| = \|x - (1-\lambda)\, Tx\| \geqslant \|x - Tx\|$ for $|\lambda|$ sufficiently small by (vi). This implies that $\lim_{\lambda \to 0^+} (\|x - Tx + \lambda Tx\| - \|x - Tx\|)/\lambda \geqslant 0$ and $\lim_{\lambda \to 0^-} (\|x - Tx + \lambda Tx\| - \|x - Tx\|)/\lambda \leqslant 0$. Hence

$$f_{x-Tx}(Tx) = 0. \tag{1}$$

By the same arguments we can deduce that

$$f_{x-Tx}(r) = 0 \qquad (r \in R). \tag{2}$$

For any $\chi_A \in TX$, again by (vi) we have

$$f_{x-Tx}(-\chi_A) = \lim_{\lambda \to 0^+} (\|x - Tx + \lambda Tx\| - \|x - Tx\|)/\lambda \geqslant 0$$

and so $f_{x-Tx}(\chi_A) \leqslant 0$.

For any $0 \leqslant u \in TX$, we have $\|u - (1/2^n) \sum_{k=0}^{n2^n} \chi_{\{u > k/2^n\}}\| \to 0$ as $n \to \infty$. Since $\{u > k/2^n\} \in \Sigma'$, $\chi_{\{u > k/2^n\}} \in TX$, we get $f_{x-Tx}(\chi_{\{u > k/2^n\}}) \leqslant 0$ $(n = 1, 2, ...; k = 1, 2, ..., n2^n)$. Furthermore, $f_{x-Tx}((1/2^n) \sum_{k=0}^{n2^n} \chi_{\{u > k/2^n\}}) \leqslant 0$, and so $f_{x-Tx}(u) \leqslant 0$.

In the general case, since $0 \leqslant u \vee (-m) + m \in TX (m = 1, 2, \ldots)$, we have $f_{x-Tx}(u \vee (-m) + m) \leqslant 0 (m = 1, 2, \ldots)$. Combining with Eq. (2), $f_{x-Tx}(u \vee (-m)) \leqslant 0 (m = 1, 2, \ldots)$. But $\|u - (u \vee (-m))\| \to 0 (m \to \infty)$. So

$$f_{x-Tx}(u) \leqslant 0 (u \in TX = X(\Sigma')). \tag{3}$$

By Eqs. (3) and (1), for any $u \in X(\Sigma')$ we have

$$\begin{aligned} \|x - Tx\| &= f_{x-Tx}(x - Tx) = f_{x-Tx}(x) \leqslant f_{x-Tx}(x - u) \\ &\leqslant \|f_{x-Tx}\|_* \, \|x - u\| = \|x - u\|. \end{aligned}$$

This means $Tx = \pi(x \mid X(\Sigma'))$.

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